Shape Design Sensitivity Analysis of Dynamic Structures

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The material derivative concept is extended for the shape sensitivity analysis of dynamic structures when the varied physical domain is time independent. Expressions are obtained by the boundary and domain methods of the sensitivity analysis. It is found that domain integrations are needed in both methods. If there are no discontinuities of the functions in a problem definition at the outset, both methods will yield the same sensitivity results, even when no discontinuity terms are considered in the boundary method of analysis.

Nomenclature

A = discontinuity surface in Ω
b_i = body forces
c = displacement wave speed, $\sqrt{E/\rho}$
D = space-time domain, $\Omega \times T$
D_m = material derivative
E = Young's modulus
H = boundary curvature
I = performance criterion
$\tilde{I}$ = augmented performance criterion
L = length of bar
L_0 = amplitude of initial displacement in bar
n_i = unit vector normal to boundary surface
t = time variable
T = interval of time $[0,t_f]$
t_f = final time
t_i = boundary tractions
t_i^* = adjoint boundary tractions
u_i = displacements
u_i^* = adjoint displacements
V_i = "deformation" velocity distribution
V_n = component of V_i normal to boundary surface
V_μ = component of V_i normal to γ and tangent to Γ
x = axial direction space variable in bar
x_i = Cartesian coordinates
x_i^τ = varied spatial coordinates, $x_i + \tau V_i$
α = ratio of velocities, ct_f/L
γ = discontinuity boundary surface curve on Γ, between Γ_1 and Γ_2
Γ = boundary surface of Ω
Γ_1 = prescribed displacements part of Γ
Γ_2 = prescribed tractions part of Γ
δL = variation in length of bar
ϵ_{ij} = strain tensor
λ = Lame's constant
λ_0 = characteristic value, $\pi/2L$
μ = Lame's constant
ρ = density
σ = normal axial stress, bar
σ_{ij} = stress tensor
σ_{ij}^* = adjoint stress tensor
τ = pseudotime parameter
ψ_1 = general integral functional, $\Omega \times T$
ψ_2 = general integral functional, $\Gamma \times T$
ω_0 = circular frequency, $c\pi/2L$
Ω = time independent physical domain to be varied
Ω^T = varied physical domain
$(\bar{\ })$ = prescribed quantity
$(\ )^0$ = initial distribution of ( )
$(\ )_{,i}$ = partial derivative of ( ) with respect to x_i
$(\dot{\ })$ = partial derivative of ( ) with respect to t
$(\ )_{,n}$ = directional derivative of ( ) normal to boundary surface
$(\ )_x$ = derivative of ( ) with respect to x
$(\ )'$ = local derivative (with respect to τ) of ( ) holding x_i constant
$[\![\ ]\!]_A$ = jump across discontinuity surface, $A \epsilon \Omega$
$[\![\ ]\!]_\gamma$ = jump across discontinuity curve, $\gamma \epsilon \Gamma$

Introduction

THERE appear to be two approaches used in the literature for optimal structural design with transient dynamic system response. In the first approach, system's equations are discretized in space (e.g., by employing finite elements) with the design specified by a finite number of parameters. Thus, in a shape sensitivity analysis (SSA), one is confronted with integral functionals and matrix ordinary differential equations defined over the time variable only (e.g., see Refs. 1-4). In the second approach, the design variables appear explicitly in the coefficients of differential operators defined over the fixed domains of interest, as in Ref. 5. A dynamic optimal shape design in which the domain shape itself is treated as the design variable may, however, prove to be more appropriate for some shape optimization or shape identification problems having transient loads or constraints.

The material derivative (MD) concept of continuum mechanics has been successfully applied to static structural optimization problems with varying domains by several authors (e.g., see Refs. 6-9). For such systems, in performing the SSA by the MD concept two alternative approaches appear to exist in the literature. In the so-called boundary method,[6-9] sensitivity information is obtained on the varying boundaries only, while the domain method[10,11] requires evaluation of domain integrals. However, it is noted here that, for continuous functions and variations in domain, differences in the SSA expressions of the two methods dissappear.

In the present paper, the MD concept (along with the adjoint variable method of optimization) is extended for SSA of dynamically loaded elastic structures. Parallel expressions for both the boundary and domain methods are obtained by using general MD formulas, adopted for the space-time domain and surface integrals. It is found that the variation of a system response integral with respect to (time-independent) domain shape variations is expressed in terms of domain as well as boundary integrals in both methods, due to the initial-boundary value character of the system equations.

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Problem Definition

Equations of motion and relevant boundary and initial conditions for a homogeneous isotropic elastic solid body in three-dimensions may be given by the following equations:

$$\text{In } \Omega, \qquad \sigma_{ij,j} + b_i = \rho \ddot{u}_i \tag{1}$$

$$\text{On } \Gamma_1, \qquad u_i = \bar{u}_i \tag{2}$$

$$\text{On } \Gamma_2, \qquad t_i = \bar{t}_i \tag{3}$$

$$\text{At } t=0, \qquad u_i = u_i^0 \tag{4}$$

$$\dot{u}_i = \dot{u}_i^0 \tag{5}$$

where Ω is the time-independent domain to be varied with its boundary $\Gamma = \Gamma_1 + \Gamma_2$ and $t_i = \sigma_{ij} n_j$ the tractions. Indicial notation with the summations indicated by repeated indices is utilized for simplicity in the analysis.

Constitutive equations describing the stress-strain relations in the body are taken as obeying Hooke's law as

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \tag{6}$$

where the strain tensor ϵ_{ij} is given by the kinematic conditions

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{7}$$

It is assumed that we are interested in the system's response within a fixed time interval $T = [0, t_f]$. A general integral performance criterion I is now defined in the space-time domain $D = \Omega \times T$ and on its "boundary," which may serve as the objective function to be minimized or a behavioral constraint to be satisfied in a shape optimization problem. That is,

$$I = \int_\Omega \int_T f(u_i, \dot{u}_i, \sigma_{ij}, \epsilon_{ij}, x_i, t) \mathrm{d}t \mathrm{d}\Omega + \int_\Gamma \int_T g(u_i, t_i, x_i, t) \mathrm{d}t \mathrm{d}\Gamma + \int_\Omega [h(u_i, \dot{u}_i, x_i)]_T \mathrm{d}\Omega \tag{8}$$

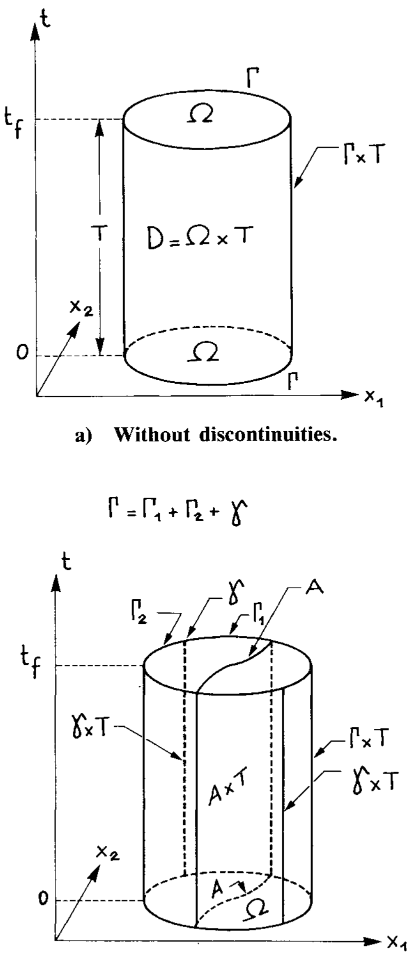
where subscript T denotes

$$[\;\cdot\;]_T = [\;\cdot\;]_{t=t_f} - [\;\cdot\;]_{t=0}$$

and f, g, and h are continuous and differentiable functions with respect to their arguments. The domain D and its boundary is depicted in Fig. 1a for two space dimensions. It is noted that since the physical domain $\Omega(x_i)$ is time independent, the domain D is represented by a perpendicular prism whose cross section is Ω at all time levels. In a moving boundary problem with $\Omega(x_i, t)$, which is not considered in the present analysis, the prism denoting D would have not been perpendicular as Ω would be changing with time.

Shape Sensitivity Analysis

Shape sensitivity analysis (SSA) of physical systems under dynamic loads may be important from different points of view: 1) to understand and model the system's behavior better with respect to shape, 2) to optimize the physical shapes of the desired system's responses in a prescribed time interval, or 3) to identify shapes by utilizing the system's measured response in time. The present SSA problem may now be stated as follows: find the total variation of I [Eq. (8)] with respect to variations in domain Ω subject to the primary problem constraints of Eqs. (1–7). SSA expressions, which should be derived for each integral functional present in, for example, an optimization problem, are then used effectively in the mathematical programming methods of minimizing a functional subject to nonlinear constraints.



a) Without discontinuities.

b) With discontinuities.

Fig. 1 Two-dimensional space-time domain $D = \Omega \times T$.

The present SSA of I for dynamically loaded elastic structures may be performed by the following steps:

1) Augment the performance criterion I by incorporating the equations of motion via adjoint functions.
2) Integrate by parts in Ω and T.
3) Take the material derivative (MD) of the augmented functional.
4) Substitute the partial derivative forms of the constitutive equations and kinematic conditions.
5) Integrate by parts again.
6) Substitute the MD forms of the boundary and initial conditions.
7) Define a suitable adjoint function.
8) Obtain the MD of I in terms of design deformation "velocity" field.

It is noted that no variational principles are needed in the above procedure, suggesting that it can be generalized to other physical systems (linear or nonlinear). Furthermore, the procedure is the same for both the boundary and domain methods of SSA. The only difference between the two methods comes from the fact that different formulas will be employed for taking the MD of integrals in step 3 of the procedure.

Material Derivatives

Before performing the SSA for the present problem by the above procedure, the MD concept[6-9] of continuum mechanics will now be extended for dynamical systems having time-independent (but varying) physical domains. Following Ref. 6, the variation of Ω under a transformation, characterized by a pseudotime parameter τ, may be regarded as a "dynamic" deformation of a continuous medium. Thus, a point x_i in Ω (at $\tau=0$) moves to the point x_i^τ in the varied domain Ω^T, given by

$$x_i^\tau = x_i + \tau V_i(x_j) \tag{9}$$

where the deformation "velocity" field V_i is time independent and is defined in the whole space, representing the rate of deformation (i.e., variation) in a SSA. The variation of Ω into Ω^τ (at all time levels) is depicted in Fig. 2.

The MD of a continuously differentiable function $w(x_i,t)$ can be given by

$$D_m w = w' + w_{,k} V_k \tag{10}$$

where $D_m w$ and w' are defined as

$$D_m w(x_i,t) = \lim_{\tau\to 0} \frac{w^\tau(x_i+\tau V_i,t) - w(x_i,t)}{\tau} \tag{11}$$

and

$$w'(x_i,t) = \lim_{\tau\to 0} \frac{w^\tau(x_i,t) - w(x_i,t)}{\tau} \tag{12}$$

respectively.[6] It is noted that partial derivatives with respect to x_i and τ commute with each other.

Before giving general formulas for the MD of the integrals, it is now convenient to introduce the integration by parts considering the discontinuity surfaces across which the functions may have jumps. Thus, for differentiable general functions u and v, the integration by parts in Ω may be written in the following form:

$$\int_\Omega uv_{,i}\mathrm{d}\Omega = \int_\Gamma uvn_i\mathrm{d}\Gamma - \int_\Omega vu_{,i}\mathrm{d}\Omega - \int_A [\![uv]\!]_A n_i \mathrm{d}A \tag{13}$$

where the quantity enclosed by the symbol $[\![\ .\ ]\!]_A$ indicates the jump across the discontinuity surface A (see Fig. 1b); that is, the difference between the quantity from the positive and negative sides of A. Hence,

$$[\![uv]\!]_A = (uv)_A^+ - (uv)_A^-$$

where $(uv)_A^+$ and $(uv)_A^-$ are the values of uv immediately near A at the positive and negative sides, respectively.[12] A similar formula for integration by parts in T may also be written, considering any discontinuities of functions in time.

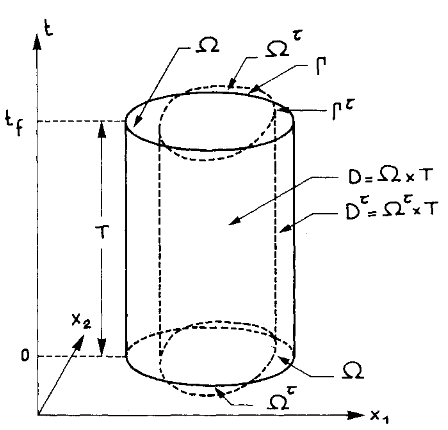


Fig. 2 Variation of space-time domain $D=\Omega\times T$ into $D^\tau=\Omega^\tau\times T$.

A general integral in $\Omega\times T$ may be defined in terms of a differentiable function $w_1(x_i,t)$ as

$$\psi_1 = \int_\Omega \int_T w_1 \mathrm{d}t\mathrm{d}\Omega \tag{14}$$

The domain method of SSA will be based on the form of $D_m\psi_1$ given as

$$D_m\psi_1 = \int_\Omega \int_T [w_1' + (w_1 V_k)_{,k}]\mathrm{d}t\mathrm{d}\Omega \tag{15}$$

It is noted that $D_m\psi_1$ involves an integral only on $\Omega\times T$ and that the deformation velocity field V_i is needed in the whole physical space Ω for calculations. Furthermore, any discontinuities in the functions w_1 and V_i will not alter the form of Eq. (15).

Using the integration by parts [Eq. (13)], Eq. (15) may also be transformed into the following formula, which is the basis of the boundary method of SSA:

$$D_m\psi_1 = \int_\Omega \int_T w_1'\mathrm{d}t\mathrm{d}\Omega + \int_\Gamma \int_T w_1 V_n \mathrm{d}t\mathrm{d}\Gamma - \int_A \int_T [\![w_1]\!]_A V_n \mathrm{d}t\mathrm{d}A \tag{16}$$

where V_n is the corresponding normal component of V_i given by

$$V_n = V_i n_i \tag{17}$$

It is now strongly emphasized that the last integral in Eq. (16) disappears if the function w_1 (and the velocity V_i) is continuous within Ω. Thus, for this case, it may be shown that the boundary and domain methods of SSA will give the same result for $D_m\psi_1$ no matter what the V_i distribution is for a given V_n on Γ.

A general integral may also be defined over $\Gamma\times T$ for smooth boundary surfaces as

$$\psi_2 = \int_\Gamma \int_T w_2 \mathrm{d}t\mathrm{d}\Gamma \tag{18}$$

The MD of ψ_2 is given in the form

$$D_m\psi_2 = \int_\Gamma \int_T [w_2' + (w_{2,n} + Hw_2)V_n]\mathrm{d}t\mathrm{d}\Gamma - \int_\gamma \int_T [\![w_2]\!]_\gamma V_\mu \mathrm{d}t\mathrm{d}\gamma \tag{19}$$

where H is the curvature of the boundary Γ in R^2 and twice the mean surface curvature of Γ in R^3. It is noted that the positive sign of H is due to the fact that n_i is taken to be positive when it is directed out of the solid body.[6] Any discontinuity of w_2 across the boundary surface curve $\gamma\epsilon\Gamma$ is taken care of by the last integral over $\gamma\times T$ in Eq. (19), where V_μ is the component of V_i on Γ, normal to γ and tangent to Γ.[13]

Boundary Method of SSA

The SSA of I [Eq. (8)] is now performed by using the proposed SSA procedure as follows:

1) The general performance criterion I is augmented by using Eq. (1) and adjoint displacements u_i^* in the following

form:

$$\tilde{I} = I + \int_\Omega \int_T u_i^*(\sigma_{ij,j} + b_i - \rho \ddot{u}_i)\mathrm{d}t\mathrm{d}\Omega \tag{20}$$

2) It will now be assumed that the solution functions and their derivatives are continuous throughout Ω and T, disregarding any discontinuities (e.g., interfaces of different media). Thus, using Eq. (8) and integrating by parts [Eq. (13)], the augmented functional $\tilde{I}$ becomes

$$\tilde{I} = \int_\Omega \int_T (f - \sigma_{ij}u_{i,j}^* + b_i u_i^* + \rho \dot{u}_i \dot{u}_i^*)\mathrm{d}t\mathrm{d}\Omega + \int_\Gamma \int_T (g + t_i u_i^*)\mathrm{d}t\mathrm{d}\Gamma + \int_\Omega [h - \rho \dot{u}_i u_i^*]_T \mathrm{d}\Omega \tag{21}$$

3) The MD of Eq. (21) is taken by using Eqs. (16) and (19). It is noted that discontinuities of only the boundary data (cf. boundary conditions) are allowed in the present SSA. The discontinuity boundary surface curve $\gamma \epsilon \Gamma$ now represents the intersection boundary curve between Γ_1 and Γ_2 (see Fig. 1b). Thus, the MD of $\tilde{I}$ is given by

$$\begin{aligned} D_m\tilde{I} = & \int_\Omega \int_T \Big[\frac{\partial f}{\partial u_i} u_i' + \Big(\frac{\partial f}{\partial \dot{u}_i} + \rho \dot{u}_i^*\Big)\dot{u}_i' - \Big(u_{i,j}^* - \frac{\partial f}{\partial \sigma_{ij}}\Big)\sigma_{ij}' \\ & + \frac{\partial f}{\partial \epsilon_{ij}}\epsilon_{ij}' - \sigma_{ij}u_{i,j}^{*\prime} + b_i u_i^{*\prime} + \rho \dot{u}_i \dot{u}_i^{*\prime}\Big]\mathrm{d}t\mathrm{d}\Omega \\ & + \int_\Gamma \int_T \Big\{[f - \sigma_{ij}u_{i,j}^* + b_i u_i^* + \rho \dot{u}_i \dot{u}_i^* + (g + t_i u_i^*)_{,n} \\ & + H(g + t_i u_i^*)]V_n + \frac{\partial g}{\partial u_i} u_i' + t_i u_i^{*\prime} + \Big(\frac{\partial g}{\partial t_i} + u_i^*\Big)t_i'\Big\}\mathrm{d}t\mathrm{d}\Gamma \\ & + \int_\Omega \Big[\frac{\partial h}{\partial u_i} u_i' + \Big(\frac{\partial h}{\partial \dot{u}_i} - \rho u_i^*\Big)\dot{u}_i' - \rho \dot{u}_i u_i^{*\prime}\Big]_T \mathrm{d}\Omega \\ & + \int_\Gamma (h - \rho \dot{u}_i u_i^*)_T V_n \mathrm{d}\Gamma - \int_\gamma \int_T [\![g + t_i u_i^*]\!]_\gamma V_\mu \mathrm{d}t\mathrm{d}\gamma \end{aligned} \tag{22}$$

4) The partial derivative (with respect to τ) forms of Eqs. (6) and (7) are simply given as

$$\sigma_{ij}' = \lambda u_{k,k}' \delta_{ij} + \mu(u_{i,j}' + u_{j,i}') \tag{23}$$

and

$$\epsilon_{ij}' = ½(u_{i,j}' + u_{j,i}') \tag{24}$$

which are directly substituted into Eq. (22).

5) Integration by parts in Ω and T is again used resulting in the following expression for $D_m\tilde{I}$:

$$\begin{aligned} D_m\tilde{I} = & \int_\Omega \int_T \Big\{(\sigma_{ij,j} + b_i - \rho \ddot{u}_i)u_i^{*\prime} + \Big[\sigma_{ij,j}^* + \frac{\partial f}{\partial u_i} - \Big(\frac{\partial f}{\partial \dot{u}_i}\Big)^{\cdot} \\ & - \rho \ddot{u}_i^*\Big]u_i'\Big\}\mathrm{d}t\mathrm{d}\Omega + \int_\Gamma \int_T \Big\{[f - \sigma_{ij}u_{i,j}^* + b_i u_i^* + \rho \dot{u}_i \dot{u}_i^* \\ & + (g + t_i u_i^*)_{,n} + H(g + t_i u_i^*)]V_n + \Big(\frac{\partial g}{\partial t_i} + u_i^*\Big)t_i' \\ & + \Big(\frac{\partial g}{\partial u_i} - t_i^*\Big)u_i'\Big\}\mathrm{d}t\mathrm{d}\Gamma + \int_\Omega \Big[\Big(\frac{\partial h}{\partial u_i} + \frac{\partial f}{\partial \dot{u}_i} + \rho \dot{u}_i^*\Big)u_i' \\ & + \Big(\frac{\partial h}{\partial \dot{u}_i} - \rho u_i^*\Big)\dot{u}_i'\Big]_T \mathrm{d}\Omega + \int_\Gamma (h - \rho \dot{u}_i u_i^*)_T V_n \mathrm{d}\Gamma \\ & - \int_\gamma \int_T [\![g + t_i u_i^*]\!]_\gamma V_\mu \mathrm{d}t\mathrm{d}\gamma \end{aligned} \tag{25}$$

where the adjoint stress tensor σ_{ij}^* is defined by

$$\sigma_{ij}^* = \lambda\Big(u_{k,k}^* - \frac{\partial f}{\partial \sigma_{kk}}\Big)\delta_{ij} + \mu\Big[\Big(u_{i,j}^* - \frac{\partial f}{\partial \sigma_{ij}}\Big) + \Big(u_{j,i}^* - \frac{\partial f}{\partial \sigma_{ji}}\Big)\Big] - \frac{\partial f}{\partial \epsilon_{ij}} \tag{26}$$

and $t_i^* = \sigma_{ij}^* n_j$.

6) By taking the MD of both sides of the boundary condition of Eq. (2) and using Eq. (10) gives

$$\text{On } \Gamma_1: \quad u_i' = \bar{u}_i' + (\bar{u}_{i,k} - u_{i,k})V_k \tag{27}$$

Following a similar procedure from Eqs. (3-5), it may be obtained that

$$\text{On } \Gamma_2: \quad t_i' = \bar{t}_i' + (\bar{t}_{i,k} - t_{i,k})V_k \tag{28}$$

$$\text{At } t=0: \quad u_i' = u_i^{0\prime} + (u_{i,k}^0 - u_{i,k})V_k \tag{29}$$

$$\dot{u}_i' = \dot{u}_i^{0\prime} + (\dot{u}_{i,k}^0 - \dot{u}_{i,k})V_k \tag{30}$$

Remark: If the prescribed quantities in Eqs. (27-30) do not depend on the shape of the domain (or its boundary), then their local variations (with respect to τ) could be set equal to zero, i.e., $\bar{u}_i' = \bar{t}_i' = u_i^{0\prime} = \dot{u}_i^{0\prime} = 0$.

The boundary surface Γ is decomposed into Γ_1 and Γ_2 parts and Eqs. (27-30) are substituted into Eq. (25) where appropriate.

7) In order to get rid of the local variations of u_i, $\dot{u}_i$, and t_i, their coefficients are set equal to zero, hence defining the following *adjoint problem*:

$$\text{In } \Omega: \quad \sigma_{ij,j}^* + \frac{\partial f}{\partial u_i} - \Big(\frac{\partial f}{\partial \dot{u}_i}\Big)^{\cdot} = \rho \ddot{u}_i^* \tag{31}$$

$$\text{On } \Gamma_1: \quad u_i^* = -\frac{\partial g}{\partial t_i} \tag{32}$$

$$\text{On } \Gamma_2: \quad t_i^* = \frac{\partial g}{\partial u_i} \tag{33}$$

$$\text{At } t = t_f: \quad u_i^* = \frac{1}{\rho}\frac{\partial h}{\partial \dot{u}_i} \tag{34}$$

$$\dot{u}_i^* = -\frac{1}{\rho}\Big(\frac{\partial h}{\partial u_i} + \frac{\partial f}{\partial \dot{u}_i}\Big) \tag{35}$$

It is noted that the adjoint problem is a final time-boundary value problem, instead of an initial time-boundary value problem, as in the case of the primary problem [Eqs. (1-7)]. In steady-state (i.e., static) problems, a physical interpretation of the adjoint equations is usually possible in terms of an "adjoint structure,"[8] for the equations represent a boundary value problem only. However, in the present dynamic case, since the adjoint equations have to be integrated backward in time from $t = t_f$ to $t = 0$, such a physical interpretation of the adjoint equations (26) and (31-35) is not directly possible.

8) When the primary and adjoint problems are satisfied for a given shape configuration of the physical domain Ω, the MD of the general performance criterion I is finally ex-

pressed as follows:

$$D_m I = \int_\Gamma \int_T [f - \sigma_{ij} u^*_{i,j} + b_i u_i^* + \rho \dot{u}_i \dot{u}_i^* + (g + t_i u_i^*)_{,n} + H(g + t_i u_i^*)] V_n \mathrm{d}t\mathrm{d}\Gamma + \int_{\Gamma_1} \int_T \left(\frac{\partial g}{\partial u_i} - t_i^*\right) \times [\bar{u}_i' + (\bar{u}_{i,k} - u_{i,k}) V_k] \mathrm{d}t\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_2} \int_T \left(\frac{\partial g}{\partial t_i} + u_i^*\right) [\bar{t}_i' + (\bar{t}_{i,k} - t_{i,k}) V_k] \mathrm{d}t\mathrm{d}\Gamma$$
$$- \int_\Omega \left\{ \left(\frac{\partial h}{\partial u_i} + \frac{\partial f}{\partial \dot{u}_i} + \rho \dot{u}_i^*\right) [u_i^{0\prime} + (u^0_{i,k} - u_{i,k}) V_k] + \left(\frac{\partial h}{\partial \dot{u}_i} - \rho u_i^*\right) [\dot{u}_i^{0\prime} + (\dot{u}_{ik} - \dot{u}_{ik}) V_k] + (\dot{u}_{i,k} - \dot{u}_{i,k}) V_k] \right\}_{t=0} \mathrm{d}\Omega$$
$$+ \int_\Gamma (h - \rho \dot{u}_i u_i^*)_T V_n \mathrm{d}\Gamma - \int_\gamma \int_T [\![g + t_i u_i^*]\!]_\gamma V_\mu \mathrm{d}t\mathrm{d}\gamma \tag{36}$$

The following remarks are made regarding the above expression:

1) Equation (36) represents the MD of I as obtained by the boundary method of SSA, disregarding any discontinuities of functions in Ω.

2) The $D_m I$ [Eq. (36)] involves "boundary" integrals only in the space-time domain $D = \Omega \times T$ (see Fig. 1a).

3) However, Eq. (36) involved an integral in the physical domain evaluated at $t = 0$. Hence, it may be said that the boundary method of SSA requires a domain integration. An assumption of the deformation velocity field V_i is also needed in the process.

4) For a simple problem at hand, the coefficients of V_i in Ω may be equal to zero (as in the example problem to be discussed later). In this case, no need arises for assuming a V_i distribution in Ω, as V_n on Γ completely describes the effects of the change in the shape of the structure.

5) For a numerical evaluation of the $D_m I$, solutions of both the primary and adjoint problems are needed, along with $V_n(x_j)$, $x_j \epsilon \Gamma$, and $V_i(x_j)$, $x_j \epsilon \Omega$, if necessary.

Domain Method of SSA

The SSA of I [Eq. (8)] can also be performed by using Eq. (15) instead of Eq. (16) for the MD of volume-time integrals. The procedure given previously also applies to the domain method of SSA, which results in the same adjoint problem [Eqs. (26) and (31-35)]. Thus, the $D_m I$ is given by the following expression in the domain method of SSA:

$$D_m I = \int_\Omega \int_T [(f - \sigma_{ij} u^*_{i,j} + b_i u_i^* + \rho \dot{u}_i \dot{u}_i^*) V_k]_{,k} \mathrm{d}t\mathrm{d}\Omega + \int_\Gamma \int_T [(g + t_i u_i^*)_{,n} + H(g + t_i u_i^*)] V_n \mathrm{d}t\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_1} \int_T \left(\frac{\partial g}{\partial u_i} - t_i^*\right) [\bar{u}_i' + (\bar{u}_{i,k} - u_{i,k}) V_k] \mathrm{d}t\mathrm{d}\Gamma + \int_{\Gamma_2} \int_T \left(\frac{\partial g}{\partial t_i} + u_i^*\right) [\bar{t}_i' + (\bar{t}_{i,k} - t_{i,k}) V_k] \mathrm{d}t\mathrm{d}\Gamma$$
$$+ \int_\Omega \{[(h - \rho \dot{u}_i u_i^*) V_k]_{,k}\}_T \mathrm{d}\Omega$$
$$- \int_\Omega \left(\frac{\partial h}{\partial u_i} + \frac{\partial f}{\partial \dot{u}_i} + \rho \dot{u}_i^*\right) [u_i^{0\prime} + (u^0_{i,k} - u_{i,k}) V_k] + \left(\frac{\partial h}{\partial \dot{u}_i} - \rho u_i^*\right) [\dot{u}_i^{0\prime} + (\dot{u}^0_{i,k} - \dot{u}_{i,k}) V_k] \Big\}_{t=0} \mathrm{d}\Omega$$
$$- \int_\gamma \int_T [\![g + t_i u_i^*]\!]_\gamma V_\mu \mathrm{d}t\mathrm{d}\gamma \tag{37}$$

Equation (37) requires domain integrations involving V_i distribution in Ω as well as V_n on Γ. It is noted that if the integration by parts [Eq. (13)] is employed, Eq. (37) reduces to Eq. (36) of the boundary method, provided no discontinuities of functions are allowed in Ω.

Example Problem

Although the present SSA of dynamically loaded elastic structures is valid for arbitrary three-dimensional shapes, a one-dimensional example might be more illustrative of the SSA expressions. Thus, as an example problem, the longitudinal free vibrations of a prismatical bar are considered. It is assumed that during the vibrations, the cross sections of the bar remain plane and the particles in these cross sections execute motion only in the axial direction. For the thin rod considered, whose cross-sectional dimensions are small, the lateral displacements are neglected. Thus, for this one-dimensional problem the equation of motion is given by[14]

$$0 < x < L: \qquad E u_{xx} = \rho \ddot{u} \tag{38}$$

where the constitutive equation has been taken as

$$\sigma = E u_x \tag{39}$$

The boundary conditions for the rod, which is built-in at one end and free at the other, are written as

$$\text{At } x = 0: \qquad u = 0 \tag{40}$$

$$\text{At } x = L: \qquad \sigma = 0 \tag{41}$$

In order to have a very simple solution for the axial displacement u, the initial conditions are chosen as

$$\text{At } t = 0: \qquad u^0 = L_0 \sin \frac{\pi x}{2L} \tag{42}$$

$$\dot{u}^0 = 0 \tag{43}$$

so that the solution for u is given in terms of the fundamental mode solution as

$$u = L_0 \sin \lambda_0 x \cos \omega_0 t \tag{44}$$

where

$$\lambda_0 = \pi/2L, \quad \omega_0 = c\lambda_0, \quad c = \sqrt{E/\rho} \tag{45}$$

The question is now to evaluate the shape sensitivity of the following integral functional I defined as

$$I = \frac{1}{2} \int_0^L \int_0^{t_f} u^2 \mathrm{d}t\mathrm{d}x \tag{46}$$

with respect to the length of the bar L.

Direct Method

It is possible to insert Eq. (44) into Eq. (46) and evaluate I as a function of L as

$$I = \frac{L_0^2 L}{8} \left(t_f + \frac{L}{c\pi} \sin \frac{c\pi t_f}{L}\right) \tag{47}$$

from which the derivative of I with respect to L is directly obtained as follows:

$$\frac{\mathrm{d}I}{\mathrm{d}L}=\frac{L_0^2}{8}\left(t_f+\frac{2L}{c\pi}\sin\frac{c\pi t_f}{L}-t_f\cos\frac{c\pi t_f}{L}\right) \tag{48}$$

The above expression can be put in the following dimensionless form as:

$$\frac{1}{L_0^2 t_f}\frac{\mathrm{d}I}{\mathrm{d}L}=\frac{1}{8}\left(1+2\,\frac{\sin\pi\alpha}{\pi\alpha}-\cos\pi\alpha\right) \tag{49}$$

where α is given by

$$\alpha=\frac{c}{(L/t_f)} \tag{50}$$

representing the ratio of "velocities." It is seen that the dimensionless quantity on the left-hand side of Eq. (49) is a function of α only.

It is noted that Eq. (49) has been obtained by first finding I as an explicit expression of L and then differentiating it. In the following, the SSA expressions of the present paper will be utilized so that the variation of I with respect to L will be found using both the boundary and domain methods of SSA, but without evaluating I explicitly in terms of L.

Adjoint Problem

The adjoint problem corresponding to I [Eq. (46)] is given by Eqs. (26) and (31-35) as

$$0<x<L: \quad Eu^*_{xx}+u=\rho\ddot{u}^* \tag{51}$$

$$\text{At } x=0: \quad u^*=0 \tag{52}$$

$$\text{At } x=L: \quad u^*_x=0 \tag{53}$$

$$\text{At } t=t_f: \quad u^*=0 \tag{54}$$

$$\dot{u}^*=0 \tag{55}$$

The solution for u^* is thus simply given as

$$u^*=\frac{L_0\sin\lambda_0 x}{4\rho\omega_0^2}\left[\cos\omega_0 t-\cos\omega_0(2t_f-t)-2\omega_0(t_f-t)\sin\omega_0 t\right] \tag{56}$$

The above solution for u^* will be used in both the boundary and domain methods of SSA, since they share the same adjoint problem in their formulations.

Boundary Method

For the SSA of I, no variation of the boundary shape will be taken at $x=0$ (i.e., $V_n=0$ at $x=0$), while the length of the bar at $x=L$ will be increased (i.e., varied) to $L+\delta L$; thus $V_n=\delta L$ at $x=L$. Using Eq. (36), it can be shown that

$$D_m I=\int_0^{t_f}(\tfrac{1}{2}u^2+\rho\dot{u}\dot{u}^*)_{x=L}\delta L\mathrm{d}t-\{[\rho\dot{u}u^*]_T\}_{x=L}\delta L - \int_0^L\{\rho\dot{u}^*[u^{0\,\prime}+(u_x^0-u_x)V]+\rho u^*\dot{u}_x V\}_{t=0}\mathrm{d}x \tag{57}$$

Introducing the initial time conditions for u and final time conditions for u^* into the above equation, it is seen that $D_m I$ takes the following form:

$$D_m I=\int_0^{t_f}(\tfrac{1}{2}u^2+\rho\dot{u}\dot{u}^*)_{x=L}\delta L\mathrm{d}t-\int_0^L(\rho\dot{u}^*u^{0\,\prime})_{t=0}\mathrm{d}x \tag{58}$$

where the coefficients of the axial perturbation velocity V have all disappeared, due to the fundamental mode solution of u in the present case. Thus, no need arises for any assumption regarding the distribution of V as a function of $x\epsilon(0,L)$.

A domain integral is still to be evaluated in Eq. (58) due to the fact that the initial displacement distribution u^0 has been taken as explicitly depending on L. The local variation of u^0 is simply given by Eq. (42) with respect to L, while holding x constant; hence,

$$u^{0\,\prime}=\left(\frac{\partial u^0}{\partial L}\right)_{x=\text{const}}\delta L=-\frac{\pi x}{2L^2}\cos\frac{\pi x}{2L}\,\delta L \tag{59}$$

Equations (44), (45), (50), (56), and (59) are now put into Eq. (58) and integrations are performed in space and time. It may be shown that the resulting expression can be written as

$$\frac{1}{L_0^2 t_f}\frac{D_m I}{\delta L}=\frac{1}{8}\left(1+2\,\frac{\sin\pi\alpha}{\pi\alpha}-\cos\pi\alpha\right) \tag{60}$$

By comparing Eqs. (49) and (60), it can be seen that the total variation of I with respect to L, represented by $D_m I/\delta L$, has been evaluated by the boundary method of SSA as exactly the same as given by the direct method.

Domain Method

For the present example problem, the SSA expression for I by the domain method [Eq. (37)] results in

$$\begin{aligned} D_m I=&\int_0^L\int_0^{t_f}[(\tfrac{1}{2}u^2-Eu_xu_x^*+\rho\dot{u}\dot{u}^*)V]_x\mathrm{d}t\mathrm{d}x \\ &+\int_0^{t_f}[E(u_xu^*)_x-Eu^*u_{xx}]_{x=L}\delta L\mathrm{d}t \\ &-\int_0^L\{[(\rho\dot{u}u^*)V]_x\}_T\mathrm{d}x \\ &-\int_0^L\{\rho\dot{u}^*[u^{0\,\prime}+(u_x^0-u_x)V]+\rho u^*\dot{u}_xV\}_{t=0}\mathrm{d}x \end{aligned} \tag{61}$$

Introducing the initial and final time conditions for u and u^*, respectively, into the above equation leads to the following equation:

$$\begin{aligned} D_m I=&\int_0^L\int_0^{t_f}[(\tfrac{1}{2}u^2-Eu_xu_x^*+\rho\dot{u}\dot{u}^*)V]_x\mathrm{d}t\mathrm{d}x \\ &+\int_0^{t_f}E(u_xu_x^*)_{x=L}\delta L\mathrm{d}t-\int_0^L(\rho\dot{u}^*u^{0\,\prime})_{t=0}\mathrm{d}x \end{aligned} \tag{62}$$

Now, however, any continuous and differentiable distribution of V in $(0,L)$, satisfying the given boundary variations at $x=0$ and L [e.g., $V=(x/L)\delta L$] will give the same expression for $D_m I$ in Eq. (62) as in the boundary method [Eq. (58)] if the integration by parts is utilized. The reason for this is due to the analytical nature of the solutions for u and u^* and the homogeneity of the elastic rod. Thus, Eq. (60) represents the SSA expression of I by the domain method of SSA as well as by the boundary method.

Conclusions

The shape sensitivity analysis (SSA) of a general integral functional I for dynamically loaded structures has been performed by the boundary and domain methods of SSA. It is found that both methods require domain integrations and that an assumption for the distribution of the deformation velocity field (which is not unique) for a given boundary perturbation of the physical domain is needed.

As long as there are no discontinuities of functions (e.g., resulting from discontinuities in material properties or body forces) in the original problem definition, the two methods of SSA give the same results, even when the discontinuity terms are disregarded in the boundary method. Any artificial discontinuities of functions due to discretizational methods[10] (e.g., finite elements) are another matter. The SSA expressions given in the paper, derived analytically, give an exact account of the total variation of I with respect to domain variations, within a first-order approximation.

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